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A stochastic description of a spin-$\frac{1}{2}$ particle in a magnetic field

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Abstract. We develop the stochastic mechanics of a non-relativistic quantum particle with spin $\frac{1}{2}$ in a possibly inhomogeneous magnetic field. We do not make any assumption on the inner structure of the particle, and we treat spin components as discrete random variables.

1. Introduction

A still open problem within Nelson's stochastic mechanics (Nelson 1966, 1967) is the description of particles with spin, in particular spin-$\frac{1}{2}$ particles.

Attempts in this direction have been based up to now on the Bopp–Haag model which interprets spinning particles as quantum rigid bodies. In this framework, to each smooth wavefunction without nodes is associated a diffusion process on the manifold $\mathbb{R}^3 \times SU(2)$ which reproduces, at any time, the quantum averages for coordinates and spin (Dankel 1970, 1977, Dohrn *et al* 1979).

A possible difficulty with this approach is due to the assumption of an extended structure for the spinning particle which is not necessary in the usual quantum theory. As is well known, this extended structure produces an additional degeneracy of quantum states corresponding to a given spin, and, in addition, the variables describing spin components at the stochastic level of the theory are continuous random variables.

Recently Faris (1982) has provided an interesting analysis indicating a mechanism which can lead to an effective discretisation of the spin components within this model.

In this paper we take a more pragmatic point of view, and while we do not pretend to construct a model of the spin, starting from the Pauli equation in a possibly inhomogeneous magnetic field $\mathcal{H}$, we associate to each smooth solution

$$\psi_t(\boldsymbol{x}) = \begin{pmatrix} \psi_t(\boldsymbol{x}, 1) \\ \psi_t(\boldsymbol{x}, -1) \end{pmatrix}$$

without zeros in $\psi_t(\boldsymbol{x}, \pm 1)$, a Markov process $\xi(t) = (\boldsymbol{x}(t), \sigma(t)) \in \mathbb{R}^3 \times \{-1, 1\}$ which reproduces quantum averages for coordinates and a selected component of the spin.

By this procedure the usual discrete structure of spin components is preserved as they are always described by discrete random variables, and the class of processes associated to wavefunctions can be characterised by suitable fields on $\mathbb{R}^3 \times \{-1, 1\}$

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which remind us of the usual osmotic and current velocities for spinless particles and obey similar non-linear differential equations of motion.

Our general approach is to start from quantum mechanics and try to interpret the continuity equation for $|\psi_t(\boldsymbol{x}, \sigma)|^2$ as a forward Kolmogorov equation†.

In order to illustrate our point of view, we consider the Schrödinger equation for a spinless particle in a magnetic field $\mathscr{H} = \text{rot}\, \boldsymbol{A}$

$$\mathrm{i}\hbar\partial_t\psi_t = \frac{1}{2m}\left(-\mathrm{i}\hbar\nabla - \frac{e}{c}\boldsymbol{A}\right)^2\psi_t + U\psi_t \tag{1.1}$$

with the associated continuity equation

$$\partial_t|\psi_t|^2 = -\frac{\hbar}{m}\,\text{div}\left[\text{Im}\,\bar{\psi}_t\left(\nabla - \frac{\mathrm{i}e}{\hbar c}\boldsymbol{A}\right)\psi_t\right]. \tag{1.2}$$

Here the problem consists of finding a vector field $\boldsymbol{b}(t, \boldsymbol{x})$ such that (1.2) takes the Fokker–Planck form

$$\partial_t\rho = \frac{\hbar}{2m}\Delta\rho - \text{div}\,\boldsymbol{b}\rho$$

which is possible if

$$\text{div}\left[|\psi_t|^2\left(\boldsymbol{b}(t, \boldsymbol{x}) - \frac{\hbar}{m}\frac{\text{Im}\,\bar{\psi}_t[\nabla - (\mathrm{i}e/\hbar c)\boldsymbol{A}]\psi_t}{|\psi_t|^2}\right)\right] = \frac{\hbar}{2m}\Delta|\psi_t|^2. \tag{1.3}$$

Of course, given ψ_t, the solution $\boldsymbol{b}(t, \boldsymbol{x})$ of (1.3) is not unique, but, under the additional restriction $\text{rot}\,\boldsymbol{b} = -(e/mc)\mathscr{H}$ which implies that $\boldsymbol{b}$ is a gradient if $e\mathscr{H} = \boldsymbol{0}$, the only possible choice is the well known solution given by Nelson:

$$\boldsymbol{b}(t, \boldsymbol{x}) = \frac{\hbar}{m}\left(\nabla\log|\psi_t| + \frac{\text{Im}\,\bar{\psi}_t[\nabla - (\mathrm{i}e/\hbar c)\boldsymbol{A}]\psi_t}{|\psi_t|^2}\right). \tag{1.4}$$

We also observe that the above procedure can be immediately generalised to density matrices ρ with smooth kernel $\rho(\boldsymbol{x}, \boldsymbol{y})$ strictly positive on the diagonal $\boldsymbol{y} = \boldsymbol{x}$ by the following general formula for the drift $\boldsymbol{b}(t, \boldsymbol{x})$:

$$\boldsymbol{b}(t, \boldsymbol{x}) = \frac{\hbar}{m}\left(\frac{1}{2}\nabla\log\rho_t(\boldsymbol{x}, \boldsymbol{x}) + \frac{\text{Im}[(\nabla_{\boldsymbol{x}} - (\mathrm{i}e/\hbar c)\boldsymbol{A})\rho_t(\boldsymbol{x}, \boldsymbol{y})]_{\boldsymbol{y}=\boldsymbol{x}}}{\rho_t(\boldsymbol{x}, \boldsymbol{x})}\right). \tag{1.5}$$

By using the expression (1.4) of $\boldsymbol{b}(t, \boldsymbol{x}) = \boldsymbol{u}(t, \boldsymbol{x}) + \boldsymbol{v}(t, \boldsymbol{x})$ and the Schrödinger equation (1.1) for ψ_t, one gets the equation of motion for the osmotic velocity

$$\boldsymbol{u} = (\hbar/m)\nabla\log|\psi_t|$$

and the current velocity

$$\boldsymbol{v} = \frac{\hbar}{m}\frac{\text{Im}\,\bar{\psi}_t[\nabla - (\mathrm{i}e/\hbar c)\boldsymbol{A}]\psi_t}{|\psi_t|^2}$$

in the forms (Nelson 1967)

$$\begin{aligned}\partial_t\boldsymbol{u} &= -(\hbar/2m)\nabla\,\text{div}\,\boldsymbol{v} - \nabla\boldsymbol{u}\cdot\boldsymbol{v} \\ \partial_t\boldsymbol{v} &= -(1/m)\nabla U + \tfrac{1}{2}\nabla|\boldsymbol{u}|^2 - \tfrac{1}{2}\nabla|\boldsymbol{v}|^2 + (\hbar/2m)\Delta\boldsymbol{u}\end{aligned} \tag{1.6}$$

† A similar approach is found in Onofri (1979).

which, of course, are equivalent to the Schrödinger equation for ψ_t under the obvious restrictions on the initial data $\boldsymbol{u}_0(\boldsymbol{x})$, $\boldsymbol{v}_0(\boldsymbol{x})$: rot $\boldsymbol{u}_0=\mathbf{0}$, rot $\boldsymbol{v}_0=-(e/mc)\boldsymbol{\mathcal{H}}$.

In the next section we shall try to follow the same scheme for a spin-$\frac{1}{2}$ particle in a constant magnetic field $\boldsymbol{\mathcal{H}}$, and only later on shall we deal with the general case of inhomogeneous $\boldsymbol{\mathcal{H}}$.

2. Constant magnetic field

In a constant magnetic field, space and spin variables are not coupled, and if

$$\psi_0(\boldsymbol{x})=\varphi_0(\boldsymbol{x})\begin{pmatrix}\chi_0(1)\\ \chi_0(-1)\end{pmatrix}$$

we can deal directly with the spin wavefunction χ_t, as the diffusion process $\boldsymbol{x}(t)$ associated to $\varphi_t(\boldsymbol{x})$ is constructed in the usual way as explained in the Introduction. We normalise to 1 the magnetic moment of the particle and call $s=(s_x, s_y, s_z)$ the vector with components given by the Pauli matrices s_α. To simplify our discussion as much as possible, we choose the usual representation of s, in which s_z is diagonal, and we decide to associate a discrete random variable $\sigma\in\{-1, 1\}$ precisely to this component of the spin.

The problem of formulating the theory in a rotationally invariant way will be discussed later.

By denoting by σ a dichotomic variable with values ±1, the equation for the spin wavefunction χ_t reads

$$\mathrm{i}\,\mathrm{d}\chi_t(\sigma)/\mathrm{d}t=\tfrac{1}{2}[\mathcal{H}_z\sigma\chi_t(\sigma)+(\mathcal{H}_x-\mathrm{i}\sigma\mathcal{H}_y)\chi_t(-\sigma)] \tag{2.1}$$

and the corresponding continuity equation for $|\chi_t(\sigma)|^2$ is

$$\mathrm{d}|\chi_t(\sigma)|^2/\mathrm{d}t=\mathrm{Im}[(\mathcal{H}_x+\mathrm{i}\sigma\mathcal{H}_y)\chi_t(\sigma)\overline{\chi_t(-\sigma)}]. \tag{2.2}$$

We now want to interpret (2.2) as the forward Kolmogorov equation for a Markov process $\sigma(t)$ with state space $Z_2=\{-1, 1\}$, which has the general form

$$\mathrm{d}\rho(t,\sigma)/\mathrm{d}t=-p(t,\sigma)\rho(t,\sigma)+p(t,-\sigma)\rho(t,-\sigma) \tag{2.3}$$

where $\rho(t,\sigma)$ is the probability distribution of $\sigma(t)$, and $p(t,\sigma)\geqslant0$ represents the jump probability per unit time from the state $\sigma\in Z_2$ to the state $-\sigma$.

By the identification $\rho(t,\sigma)=|\chi_t(\sigma)|^2$, our problem now consists of finding an expression for $p(t,\sigma)$ in terms of $\chi_t(\sigma)$ in such a way that

$$\mathrm{Im}[(\mathcal{H}_x+\mathrm{i}\sigma\mathcal{H}_y)\chi_t(\sigma)\overline{\chi_t(-\sigma)}]=-p(t,\sigma)|\chi_t(\sigma)|^2+p(t,-\sigma)|\chi_t(-\sigma)|^2. \tag{2.4}$$

It is easy to see that the general solution of (2.4) is given by

$$p(t,\sigma)=\frac{1}{2}\left[\phi(t)\left|\frac{\chi_t(-\sigma)}{\chi_t(\sigma)}\right|+\mathrm{Im}\left((\mathcal{H}_x-\mathrm{i}\sigma\mathcal{H}_y)\frac{\chi_t(-\sigma)}{\chi_t(\sigma)}\right)\right]. \tag{2.5}$$

In (2.5) $\phi(t)$ may depend on $\chi_t(\sigma)$, but only through combinations independent of σ and insensitive to an overall phase factor. The only possibility then is $\phi(t)=\phi(\rho(t,\sigma)\cdot\rho(t,-\sigma))$. In order to complete our construction we must give a criterion which fixes ϕ. At this point we observe that our procedure depends only on the form of equations (2.2), (2.3) but not on the normalisation of $\rho(t,\sigma)=|\chi_t(\sigma)|^2$, and it is rather natural to require that $p(t, \sigma)$ be invariant (as the drift $(\boldsymbol{b}(t, \boldsymbol{x})$ in the usual Nelson theory) under the scale transformation $\chi_t \to \lambda\chi_t$. The only way of doing that is to choose ϕ as a constant independent of χ_t, and the minimal choice, under the positivity condition $p(t, \sigma) \geq 0$, is

$$\phi = |\mathscr{H}_x \pm i\mathscr{H}_y| = (\mathscr{H}_x^2 + \mathscr{H}_y^2)^{1/2}.$$

In the next section we shall show that the constant $(\mathscr{H}_x^2 + \mathscr{H}_y^2)^{1/2}$ is actually the correct one by an independent study of the ground state process.

At this point we have completed the construction of the process $\sigma(t)$ associated to χ_t by the formulae

$$\rho(0, \sigma) = |\chi_0(\sigma)|^2 \tag{2.6}$$

$$p(t, \sigma) = \frac{1}{2}\left[(\mathscr{H}_x^2 + \mathscr{H}_y^2)^{1/2}\left|\frac{\chi_t(-\sigma)}{\chi_t(\sigma)}\right| + \operatorname{Im}\left((\mathscr{H}_x - i\sigma\mathscr{H}_y)\frac{\chi_t(-\sigma)}{\chi_t(\sigma)}\right)\right]. \tag{2.7}$$

We now want to characterise $\sigma(t)$ in terms of suitable non-linear differential equations by an analogy with (1.6).

By defining $\mathfrak{r}(t, \sigma)$, $\mathfrak{s}(t, \sigma)$ as

$$\mathfrak{r}(t, \sigma) = \mathscr{H}_z + \sigma \operatorname{Re}\left((\mathscr{H}_x - i\sigma\mathscr{H}_y)\frac{\chi_t(-\sigma)}{\chi_t(\sigma)}\right) \tag{2.8}$$

$$\mathfrak{s}(t, \sigma) = -\sigma \operatorname{Im}\left((\mathscr{H}_x - i\sigma\mathscr{H}_y)\frac{\chi_t(-\sigma)}{\chi_t(\sigma)}\right) \tag{2.9}$$

we get, from (2.1), the equations of motion

$$\begin{aligned} \mathrm{d}\mathfrak{r}(t, \sigma)/\mathrm{d}t &= -\sigma\mathfrak{r}(t, \sigma)\mathfrak{s}(t, \sigma) \\ \mathrm{d}\mathfrak{s}(t, \sigma)/\mathrm{d}t &= -\tfrac{1}{2}\sigma|\boldsymbol{\mathscr{H}}|^2 + \tfrac{1}{2}\sigma\mathfrak{r}^2(t, \sigma) - \tfrac{1}{2}\sigma\mathfrak{s}^2(t, \sigma). \end{aligned} \tag{2.10}$$

It is easy to see that the processes constructed according to the rule (2.6), (2.7) can also be defined as the processes with probability distribution $\rho(t, \sigma)$, and jump probability $p(t, \sigma)$ given by

$$\begin{pmatrix} \mathfrak{r}(t, \sigma) - \mathscr{H}_z & \mathfrak{r}(t, -\sigma) - \mathscr{H}_z \\ \mathfrak{s}(t, \sigma) & \mathfrak{s}(t, -\sigma) \end{pmatrix}\begin{pmatrix} \rho(t, \sigma) \\ \rho(t, -\sigma) \end{pmatrix} = 0 \qquad \rho(t, \sigma) + \rho(r, -\sigma) = 1 \tag{2.11}$$

$$p(t, \sigma) = \tfrac{1}{2}\{[\mathfrak{s}^2 + (\mathfrak{r} - \mathscr{H}_z)^2]^{1/2} - \sigma\mathfrak{s}(t, \sigma)\} \tag{2.12}$$

where $\mathfrak{r}(t, \sigma)$, $\mathfrak{s}(t, \sigma)$ is any solution of the non-linear equations (2.10) with the following restrictions on the initial data:

$$\begin{aligned} \mathfrak{s}_0(\sigma)\mathfrak{r}_0(-\sigma) + \mathfrak{s}_0(-\sigma)\mathfrak{r}_0(\sigma) &= \mathscr{H}_z(\mathfrak{s}_0(\sigma) + \mathfrak{s}_0(-\sigma)) \\ \mathfrak{r}_0(\sigma)\mathfrak{r}_0(-\sigma) - \mathfrak{s}_0(\sigma)\mathfrak{s}_0(-\sigma) &= \mathscr{H}_z(\mathfrak{r}_0(\sigma) + \mathfrak{r}_0(-\sigma)) - |\boldsymbol{\mathscr{H}}|^2. \end{aligned} \tag{2.13}$$

These, of course, are equivalent to (2.8) and (2.9) for a suitable $\chi_0(\sigma)$ defined up to an overall phase factor and a normalisation scale. Notice that in (2.10) the different values of σ are coupled only through the initial conditions (2.13).

In the particular case $\boldsymbol{\mathscr{H}} = (\omega, 0, 0)$, the quantum system (2.1) is formally equivalent to a Fermi oscillator with Hamiltonian

$$H = \hbar\omega a^* a = \tfrac{1}{4}\hbar\omega(s_z + is_y)(s_z - is_y).$$

The resulting stochastic theory for the coordinate $s_z = a + a^*$ is a little bit different from the one developed in De Angelis *et al* (1981) except for stationary states.

3. Stationary solutions

Now we consider stationary solutions of (2.10) in the non-trivial case $\mathscr{H}_x^2 + \mathscr{H}_y^2 > 0$ where, by taking into account (2.13), we find

$$\delta^{\pm}(\sigma) = 0 \qquad \imath^{\pm}(\sigma) = \pm\sigma|\boldsymbol{\mathscr{H}}| \tag{3.1}$$

which correspond to the two stationary processes $\sigma^{\pm}(t)$ defined by

$$\rho^{\pm}(\sigma) = \frac{|\boldsymbol{\mathscr{H}}| \pm \sigma\mathscr{H}_z}{2|\boldsymbol{\mathscr{H}}|} \qquad p^{\pm}(\sigma) = \frac{|\boldsymbol{\mathscr{H}}| \mp \sigma\mathscr{H}_z}{2}. \tag{3.2}$$

According to (2.6), (2.7) we interpret $\sigma^+(t)$ as the ground state processes and $\sigma^-(t)$ as the excited state process. We can define $\sigma^{\pm}(t)$ by deriving the corresponding transition probabilities $P_t^{\pm}(\sigma, \sigma')$ from the Kolmogorov equation (2.3) to which are associated the generators $L^{\pm}$:

$$(L^{\pm}f)(\sigma) = p^{\pm}(\sigma)(f(-\sigma) - f(\sigma)). \tag{3.3}$$

In this way we can find the correlation functions, for instance

$$\begin{aligned} &\mathbb{E}(\sigma^{\pm}(t)) = \pm\mathscr{H}_z/|\boldsymbol{\mathscr{H}}| \\ &\mathbb{E}(\sigma^{\pm}(0)\sigma^{\pm}(t)) = \mathrm{e}^{-t|\boldsymbol{\mathscr{H}}|} + (\mathscr{H}_z^2/|\boldsymbol{\mathscr{H}}|^2)(1 - \mathrm{e}^{-t|\boldsymbol{\mathscr{H}}|}) \qquad t \geq 0. \end{aligned} \tag{3.4}$$

If $\mathscr{H}_z = 0$ we recognise that $\sigma^{\pm}(t)$ becomes a Poisson process and coincides with the ground state process described in De Angelis *et al* (1981).

A check on the consistency of our scheme is provided by the following independent construction of the ground state process. Since the ground state wavefunction $\Omega(\cdot)$ does not exhibit zeros for $\mathscr{H}_x^2 + \mathscr{H}_y^2 > 0$, we can perform the well known unitary transformation (see Reed and Simon (1975) for elementary examples, and more generally Albeverio *et al* (1977))

$$\chi(\sigma) \to \Omega^{-1}(\sigma)\chi(\sigma) \tag{3.5}$$

which carries $\Omega(\cdot)$ into the constant unit function $\mathbb{1}$.

Under this map the Hamiltonian $H = -\frac{1}{2}\boldsymbol{\mathscr{H}} \cdot s - E_0$, with the ground state energy E_0 subtracted, becomes the Kolmogorov operator $-L$ given by

$$(-Lf)(\sigma) = \tfrac{1}{2}(|\boldsymbol{\mathscr{H}}| - \sigma\mathscr{H}_z)(f(\sigma) - f(-\sigma)) \tag{3.6}$$

which coincides with the previously defined L^+ and is the generator of a Markovian semigroup

$$(\mathrm{e}^{tL}f)(\sigma) = \sum_{\sigma'=\pm 1} P_t^+(\sigma, \sigma')f(\sigma') \tag{3.7}$$

where

$$P_t(\sigma, \sigma') = \frac{|\boldsymbol{\mathscr{H}}|(1 + \sigma\sigma'\,\mathrm{e}^{-t|\boldsymbol{\mathscr{H}}|}) + \mathscr{H}_z\sigma'(1 - \mathrm{e}^{-t|\boldsymbol{\mathscr{H}}|})}{2|\boldsymbol{\mathscr{H}}|} \tag{3.8}$$

is the transition probability of the process $\sigma^+(t)$.

4. Invariance under the rotation group

We end the discussion of the homogenous magnetic field case by removing the previous choice of a particular component of the spin, namely s_z, as random variable; moreover, we generalise the theory by considering also non-pure states in spin space.

Let $\langle\cdot\rangle_t$ be any state, pure or not, in the spin space and $\boldsymbol{n}$ some unit vector.

According to quantum mechanics, the probability distribution of the observable $\boldsymbol{n}\cdot\boldsymbol{s}$ at time t is given by

$$\rho(t,\sigma)=\tfrac{1}{2}\langle 1+\boldsymbol{n}\cdot\boldsymbol{s}\rangle_t \tag{4.1}$$

which obeys the continuity equation

$$\mathrm{d}\rho(t,\sigma)/\mathrm{d}t=\tfrac{1}{2}\sigma\boldsymbol{\mathcal{H}}\times\boldsymbol{n}\cdot\langle\boldsymbol{s}\rangle_t. \tag{4.2}$$

By repeating the steps of § 2, we transform (4.2) in the Kolmogorov equation (2.4) with the general choice

$$p(t,\sigma)=\tfrac{1}{2}\left[\phi(t)\left(\frac{\rho(t,-\sigma)}{\rho(t,\sigma)}\right)^{1/2}-\frac{\sigma\boldsymbol{\mathcal{H}}\times\boldsymbol{n}\cdot\langle\boldsymbol{s}\rangle_t}{2\rho(t,\sigma)}\right] \tag{4.3}$$

and try to take ϕ as a constant independent of the particular quantum state $\langle\cdot\rangle$. The minimal choice which assures the positivity of $p(t,\sigma)$ is

$$\phi=|\boldsymbol{\mathcal{H}}\times\boldsymbol{n}| \tag{4.4}$$

and with this particular choice of ϕ, the jump probability per unit time $p(t,\sigma)$ and the probability distribution $\rho(t,\sigma)$ become

$$\rho(t,\sigma|\langle\cdot\rangle,\boldsymbol{\mathcal{H}},\boldsymbol{n})=\tfrac{1}{2}\langle 1+\boldsymbol{n}\cdot\boldsymbol{s}\rangle_t \tag{4.5}$$

$$p(t,\sigma|\langle\cdot\rangle,\boldsymbol{\mathcal{H}},\boldsymbol{n})=\frac{1}{2}\left[|\boldsymbol{\mathcal{H}}\times\boldsymbol{n}|\left(\frac{\langle 1-\sigma\boldsymbol{n}\cdot\boldsymbol{s}\rangle_t}{\langle 1+\sigma\boldsymbol{n}\cdot\boldsymbol{s}\rangle_t}\right)^{1/2}-\sigma\frac{\boldsymbol{\mathcal{H}}\times\boldsymbol{n}\cdot\langle\boldsymbol{s}\rangle_t}{\langle 1+\sigma\boldsymbol{n}\cdot\boldsymbol{s}\rangle_t}\right] \tag{4.6}$$

which, of course, reduce to (2.7), (2.8) for a pure state and in the special case $\boldsymbol{n}=(0,0,1)$. If U is an element of SU(2) and $R(U)$ the associated matrix in SO(3) by $U^{-1}\boldsymbol{s}U=R(U)\boldsymbol{s}$, it is immediately seen that a simultaneous rotation of the state $\langle\cdot\rangle$, the magnetic field $\boldsymbol{\mathcal{H}}$ and the unit vector $\boldsymbol{n}$

$$\langle\cdot\rangle\to\langle\cdot\rangle^U,\qquad \boldsymbol{\mathcal{H}}\to R(U)\boldsymbol{\mathcal{H}},\qquad \boldsymbol{n}\to R(U)\boldsymbol{n} \tag{4.7}$$

leaves the process (4.5), (4.6) unchanged. In this manner we see that our stochastic construction is rotationally invariant according to the isotropy of space.

In the general case the quantities $\mathscr{r}(t,\sigma)$, $\mathscr{s}(t,\sigma)$ of § 2 are replaced by

$$\mathscr{r}(t,\sigma)=\boldsymbol{n}\cdot\boldsymbol{\mathcal{H}}+\sigma\frac{\boldsymbol{\mathcal{H}}^{\perp}\cdot\langle\boldsymbol{s}\rangle_t}{\langle 1+\sigma\boldsymbol{n}\cdot\boldsymbol{s}\rangle_t}\qquad \boldsymbol{\mathcal{H}}^{\perp}=\boldsymbol{\mathcal{H}}-(\boldsymbol{n}\cdot\boldsymbol{\mathcal{H}})\boldsymbol{n} \tag{4.8}$$

$$\mathscr{s}(t,\sigma)=\frac{\boldsymbol{\mathcal{H}}\times\boldsymbol{n}\cdot\langle\boldsymbol{s}\rangle_t}{\langle 1+\sigma\boldsymbol{n}\cdot\boldsymbol{s}\rangle_t} \tag{4.9}$$

and, if $\langle\cdot\rangle$ is a pure state, still obey the equations (2.10).

The new restrictions on the initial data of (2.10) are, of course,

$$\begin{aligned}&\mathscr{s}_0(\sigma)\mathscr{r}_0(-\sigma)+\mathscr{s}_0(-\sigma)\mathscr{r}_0(\sigma)=\boldsymbol{n}\cdot\boldsymbol{\mathcal{H}}(\mathscr{s}_0(\sigma)+\mathscr{s}_0(-\sigma))\\&\mathscr{r}_0(\sigma)\mathscr{r}_0(-\sigma)-\mathscr{s}_0(\sigma)\mathscr{s}_0(-\sigma)=\boldsymbol{n}\cdot\boldsymbol{\mathcal{H}}(\mathscr{r}_0(\sigma)+\mathscr{r}_0(-\sigma))-|\boldsymbol{\mathcal{H}}|^2\end{aligned} \tag{4.10}$$

while the Markov process associated to a solution of (2.10), (4.10) is defined by

$$p(t,\sigma)=\tfrac{1}{2}\{[\jmath^2+(\imath-\boldsymbol{n}\cdot\boldsymbol{\mathcal{H}})^2]^{1/2}-\sigma\jmath(t,\sigma)\} \tag{4.11}$$

$$\begin{pmatrix}\imath(t,\sigma)-\boldsymbol{n}\cdot\boldsymbol{\mathcal{H}} & \imath(t,\sigma)-\boldsymbol{n}\cdot\boldsymbol{\mathcal{H}}\\ \jmath(t,\sigma) & \jmath(t,-\sigma)\end{pmatrix}\begin{pmatrix}\rho(t,\sigma)\\ \rho(t,-\sigma)\end{pmatrix}=0 \qquad \rho(t,\sigma)+\rho(t,-\sigma)=1 \tag{4.12}$$

which generalise (2.11), (2.12). The dependence of $\imath$ and $\jmath$ on σ and $\boldsymbol{n}$ is again introduced by the initial conditions.

5. Inhomogeneous magnetic field

Starting from the Pauli equation

$$\mathrm{i}\hbar\partial_t\psi_t=\frac{1}{2m}\left(-\mathrm{i}\hbar\nabla-\frac{e}{c}\boldsymbol{A}\right)^2\psi_t+U\psi_t-\frac{\hbar}{2}\boldsymbol{\mathcal{H}}\cdot\boldsymbol{s}\psi_t \tag{5.1}$$

and introducing the joint probability distribution of $\boldsymbol{x}$ and $\boldsymbol{n}\cdot\boldsymbol{s}$

$$\rho^\sigma(t,\boldsymbol{x})=\tfrac{1}{2}\langle\psi_t(\boldsymbol{x}),(1+\sigma\boldsymbol{n}\cdot\boldsymbol{s})\psi_t(\boldsymbol{x})\rangle \tag{5.2}$$

the following continuity equation holds for ρ:

$$\partial_t\rho^\sigma=-\frac{\hbar}{m}\operatorname{div}\operatorname{Im}\left\langle\psi_t(\boldsymbol{x}),\frac{1+\sigma\boldsymbol{n}\cdot\boldsymbol{s}}{2}\left(\nabla-\frac{\mathrm{i}e}{\hbar c}\boldsymbol{A}\right)\psi_t(\boldsymbol{x})\right\rangle+\frac{\sigma}{2}\boldsymbol{\mathcal{H}}\times\boldsymbol{n}\cdot\langle\psi_t(\boldsymbol{x}),\boldsymbol{s}\psi_t(\boldsymbol{x})\rangle. \tag{5.3}$$

This equation takes the Kolmogorov form

$$\partial_t\rho^\sigma=(\hbar/2m)\Delta\rho^\sigma-\operatorname{div}\boldsymbol{b}^\sigma\rho^\sigma-p^\times(t,\sigma)\rho^\sigma+p^\times(t,-\sigma)\rho^{-\sigma} \tag{5.4}$$

with $p^\times(t,\sigma)\geq 0$, if $\boldsymbol{b}^\sigma(t,\boldsymbol{x})$ and $p^\times(t,\sigma)$ are defined by

$$\boldsymbol{b}^\sigma(t,\boldsymbol{x})=\frac{\hbar}{m}\left(\frac{1}{2}\nabla\log\rho^\sigma+\frac{\operatorname{Im}\langle\psi_t(\boldsymbol{x}),(1+\sigma\boldsymbol{n}\cdot\boldsymbol{s})[\nabla-(\mathrm{i}e/\hbar c)\boldsymbol{A}]\psi_t(\boldsymbol{x})\rangle}{\langle\psi_t(\boldsymbol{x}),(1+\sigma\boldsymbol{n}\cdot\boldsymbol{s})\psi_t(\boldsymbol{x})\rangle}\right) \tag{5.5}$$

$$p^\times(t,\sigma)=\frac{1}{2}\left[|\boldsymbol{\mathcal{H}}\times\boldsymbol{n}|\left(\frac{\rho^{-\sigma}(t,\boldsymbol{x})}{\rho^\sigma(t,\boldsymbol{x})}\right)^{1/2}-\sigma\frac{\boldsymbol{\mathcal{H}}\times\boldsymbol{n}\cdot\langle\psi_t(\boldsymbol{x}),\boldsymbol{s}\psi_t(\boldsymbol{x})\rangle}{\langle\psi_t(\boldsymbol{x}),(1+\sigma\boldsymbol{n}\cdot\boldsymbol{s})\psi_t(\boldsymbol{x})\rangle}\right]. \tag{5.6}$$

Of course, if $\boldsymbol{\mathcal{H}}$ is homogeneous and $\psi_t(\boldsymbol{x},\sigma)=\varphi_t(\boldsymbol{x})\chi_t(\sigma)$,

$$\boldsymbol{b}^\sigma(t,\boldsymbol{x})=\boldsymbol{b}(t,\boldsymbol{x})=\frac{\hbar}{m}\left(\nabla\log|\varphi_t|+\frac{\operatorname{Im}\bar{\varphi}_t[\nabla-(\mathrm{i}e/\hbar c)\boldsymbol{A}]\varphi_t}{|\varphi_t|^2}\right)$$

and

$$p^\times(t,\sigma)=p(t,\sigma)=\frac{1}{2}\left[|\boldsymbol{\mathcal{H}}\times\boldsymbol{n}|\left(\frac{\langle\chi_t,(1-\sigma\boldsymbol{n}\cdot\boldsymbol{s})\chi_t\rangle}{\langle\chi_t,(1+\sigma\boldsymbol{n}\cdot\boldsymbol{s})\chi_t\rangle}\right)^{1/2}-\sigma\frac{\boldsymbol{\mathcal{H}}\times\boldsymbol{n}\cdot\langle\chi_t,\boldsymbol{s}\chi_t\rangle}{\langle\chi_t,(1+\sigma\boldsymbol{n}\cdot\boldsymbol{s})\chi_t\rangle}\right].$$

The Kolmogorov equation (5.4) then splits into two separate pieces, and we recover the ordinary Nelson construction and the stochastic theory of spin developed in §§ 2–4.

In the general case when (5.4) does not split, the two processes $\boldsymbol{x}(t)$ and $\sigma(t)$ in $\xi(t)=(\boldsymbol{x}(t),\sigma(t))\in\mathbb{R}^3\times\{-1,1\}$ are not independent, but the whole process $\xi(t)$ can be defined by constructing its transition probability as the fundamental solution of (5.4) which is connected with the Kolmogorov operator

$$(L_t f)(\boldsymbol{x},\sigma)=\frac{\hbar}{2m}\Delta f+\boldsymbol{b}^\sigma\cdot\nabla f+p^\times(t,\sigma)(f(\boldsymbol{x},-\sigma)-f(\boldsymbol{x},\sigma)) \tag{5.7}$$

together with the initial probability distribution represented by

$$\rho_0^\sigma(\boldsymbol{x}) = \tfrac{1}{2}\langle\psi_0(\boldsymbol{x}), (1+\sigma\boldsymbol{n}\cdot\boldsymbol{s})\psi_0(\boldsymbol{x})\rangle$$

if $\psi_0(\cdot)$ is correctly normalised.

As in the Introduction, we observe that the previous construction is by no means restricted to wavefunctions and can be immediately generalised to smooth density matrices $\rho_t(\boldsymbol{x},\boldsymbol{y}) \equiv \|\rho_t^{\sigma\sigma'}(\boldsymbol{x},\boldsymbol{y})\|$ verifying the condition of positivity $\mathrm{Tr}[(1+\sigma\boldsymbol{n}\cdot\boldsymbol{s})\rho_t(\boldsymbol{x},\boldsymbol{x})] > 0$, by the following natural extensions of (5.5), (5.6):

$$\begin{aligned}
\boldsymbol{b}^\sigma(t,\boldsymbol{x}) &= \frac{\hbar}{m}\left(\frac{1}{2}\nabla\log\rho^\sigma + \frac{\mathrm{Im}[[\nabla_x - (\mathrm{i}e/\hbar c)\boldsymbol{A}]\,\mathrm{Tr}[(1+\sigma\boldsymbol{n}\cdot\boldsymbol{s})\rho_t(\boldsymbol{x},\boldsymbol{y})]]_{\boldsymbol{y}=\boldsymbol{x}}}{2\rho^\sigma}\right) \\
p^{\times}(t,\sigma) &= \frac{1}{2}\left[|\boldsymbol{\mathcal{H}}(\boldsymbol{x})\times\boldsymbol{n}|\left(\frac{\rho^{-\sigma}(t,\boldsymbol{x})}{\rho^\sigma(t,\boldsymbol{x})}\right)^{1/2} - \sigma\frac{\mathrm{Tr}(\rho_t(\boldsymbol{x},\boldsymbol{x})\boldsymbol{\mathcal{H}}\times\boldsymbol{n}\cdot\boldsymbol{s})}{2\rho^\sigma(t,\boldsymbol{x})}\right].
\end{aligned} \tag{5.8}$$

Here, of course,

$$\rho^\sigma(t,\boldsymbol{x}) = \tfrac{1}{2}\mathrm{Tr}[\rho_t(\boldsymbol{x},\boldsymbol{x})(1+\sigma\boldsymbol{n}\cdot\boldsymbol{s})]. \tag{5.9}$$

In the pure state case it is possible to characterise the processes constructed according to (5.5), (5.6) by suitable equations of motion for the 'velocity fields' $\boldsymbol{u}^\sigma(t,\boldsymbol{x})$, $\boldsymbol{v}^\sigma(t,\boldsymbol{x})$, $\imath^{\times}(t,\sigma)$, $\jmath^{\times}(t,\sigma)$ on $\mathbb{R}^3\times\{-1,1\}$ defined according to

$$\begin{aligned}
\boldsymbol{u}^\sigma(t,\boldsymbol{x}) &= (\hbar/2m)\nabla\log\rho^\sigma \\
\boldsymbol{v}^\sigma(t,\boldsymbol{x}) &= \frac{\hbar}{m}\frac{\mathrm{Im}\langle\psi_t(\boldsymbol{x}),(1+\sigma\boldsymbol{n}\cdot\boldsymbol{s})[\nabla-(\mathrm{i}e/\hbar c)\boldsymbol{A}]\psi_t(\boldsymbol{x})\rangle}{\langle\psi_t(\boldsymbol{x}),(1+\sigma\boldsymbol{n}\cdot\boldsymbol{s})\psi_t(\boldsymbol{x})\rangle} \\
\imath^{\times}(t,\sigma) &= \boldsymbol{n}\cdot\boldsymbol{\mathcal{H}} + \sigma\frac{\boldsymbol{\mathcal{H}}^\perp\cdot\langle\psi_t(\boldsymbol{x}),\boldsymbol{s}\psi_t(\boldsymbol{x})\rangle}{\langle\psi_t(\boldsymbol{x}),(1+\sigma\boldsymbol{n}\cdot\boldsymbol{s})\psi_t(\boldsymbol{x})\rangle} \\
\jmath^{\times}(t,\sigma) &= \frac{\boldsymbol{\mathcal{H}}\times\boldsymbol{n}\cdot\langle\psi_t(\boldsymbol{x}),\boldsymbol{s}\psi_t(\boldsymbol{x})\rangle}{\langle\psi_t(\boldsymbol{x}),(1+\sigma\boldsymbol{n}\cdot\boldsymbol{s})\psi_t(\boldsymbol{x})\rangle}
\end{aligned} \tag{5.10}$$

in terms of which one can reconstruct $\rho^\sigma(t,\boldsymbol{x})$, $\boldsymbol{b}^\sigma(t,\boldsymbol{x})$ and $p^{\times}(t,\sigma)$. In order to derive such equations it is expedient first to use the special choice $\boldsymbol{n} = (0,0,1)$ and then exploit the rotational invariance of the theory. By rewriting $\psi_t(\boldsymbol{x},\sigma)$ as $\exp(R_t^\sigma(\boldsymbol{x}) + \mathrm{i}S_t^\sigma(\boldsymbol{x}))$ and proceeding as in Nelson (1967), it is not difficult to see that

$$\begin{aligned}
\partial_t\boldsymbol{u}^\sigma &= -(\hbar/2m)\nabla\,\mathrm{div}\,\boldsymbol{v}^\sigma - \nabla\boldsymbol{u}^\sigma\cdot\boldsymbol{v}^\sigma + \sigma(\hbar/2m)\nabla\jmath^{\times} \\
\partial_t\boldsymbol{v}^\sigma &= -(1/m)\nabla U + \tfrac{1}{2}\nabla|\boldsymbol{u}^\sigma|^2 - \tfrac{1}{2}\nabla|\boldsymbol{v}^\sigma|^2 + (\hbar/2m)\Delta\boldsymbol{u}^\sigma + \sigma(\hbar/2m)\nabla\imath^{\times}, \\
\partial_t\imath^{\times} &= -\sigma\imath^{\times}\jmath^{\times} + (\imath^{\times} - \boldsymbol{n}\cdot\boldsymbol{\mathcal{H}})\partial_\sigma[\tfrac{1}{2}\,\mathrm{div}\,\boldsymbol{v}^\sigma + (m/\hbar)\boldsymbol{u}^\sigma\cdot\boldsymbol{v}^\sigma] \\
&\quad - \jmath^{\times}\partial_\sigma[\tfrac{1}{2}\,\mathrm{div}\,\boldsymbol{u}^\sigma + (m/2\hbar)(|\boldsymbol{u}^\sigma|^2 - |\boldsymbol{v}^\sigma|^2)] \\
\partial_t\jmath^{\times} &= -\tfrac{1}{2}\sigma|\boldsymbol{\mathcal{H}}|^2 + \tfrac{1}{2}\sigma(\imath^{\times})^2 - \tfrac{1}{2}\sigma(\jmath^{\times})^2 + \jmath^{\times}\partial_\sigma[\tfrac{1}{2}\,\mathrm{div}\,\boldsymbol{v}^\sigma + (m/\hbar)\boldsymbol{u}^\sigma\cdot\boldsymbol{v}^\sigma] \\
&\quad + (\imath^{\times} - \boldsymbol{n}\cdot\boldsymbol{\mathcal{H}})\partial_\sigma[\tfrac{1}{2}\,\mathrm{div}\,\boldsymbol{u}^\sigma + (m/2\hbar)(|\boldsymbol{u}^\sigma|^2 - |\boldsymbol{v}^\sigma|^2)]
\end{aligned} \tag{5.11}$$

where we use $\partial_\sigma f$ as a short notation for $f(\boldsymbol{x},\sigma) - f(\boldsymbol{x},-\sigma)$.

The equations (5.11) must be solved under the restrictions (4.10) and rot $\boldsymbol{u}_0^\sigma = \boldsymbol{0}$, rot $\boldsymbol{v}_0^\sigma = -(e/mc)\boldsymbol{\mathcal{H}}$ on the initial data $\boldsymbol{u}_0^\sigma(x)$, $\boldsymbol{v}_0^\sigma(x)$, $\imath_0^{\times}(\sigma)$, $\jmath_0^{\times}(\sigma)$ and, of course, are equivalent to the Pauli equation for ψ_t.

As a last remark we observe that, if $\mathcal{H}(x)/|\mathcal{H}(x)|$ is a constant unit vector and n is chosen as $\mathcal{H}(x)/|\mathcal{H}(x)|$, the equations for $\imath^x(t,\sigma)$ and $\jmath^x(t,\sigma)$ are solved by $\jmath^x(t,\sigma)=0$ and $\imath^x(t,\sigma)=|\mathcal{H}(x)|$.

The resulting equations for $u^\sigma(t,x)$, $v^\sigma(t,x)$

$$\begin{aligned}\partial_t u^\sigma &= -(\hbar/2m)\nabla\,\mathrm{div}\,v^\sigma - \nabla u^\sigma\cdot v^\sigma\\ \partial_t v^\sigma &= -(1/m)\nabla[U-(\sigma\hbar/2)|\mathcal{H}|]+\tfrac{1}{2}\nabla|u^\sigma|^2-\tfrac{1}{2}\nabla|v^\sigma|^2+(\hbar/2m)\Delta u^\sigma\end{aligned}\tag{5.12}$$

are decoupled in σ and describe two independent diffusion processes $x^\pm(t)$ clearly related to the whole process $\xi(t)=(x(t),\sigma(t))$ by $x^\pm(t)=\mathbb{E}(x(t)|\sigma(t)=\pm1)$, where $\mathbb{E}(\cdot|\sigma(t)=\pm1)$ is the conditional expectation with respect to the spin process $\sigma(t)$ which, in turn, is trivial because $p^x(t,\sigma)=0$.

This is an interesting case as it covers, for example, Stern–Gerlach-type experiments.

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